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Generalized Gaussian Birkhoff Quadrature Formulas*

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Existence of a generalized Gaussian Birkhoff quadrature formula is proved for a wide class of incidence matrices which satisfy the delayed Pólya conditions and contain no odd non-Hermitian sequences in the interior rows. © 1995 Academic Press, Inc.

This paper deals with the existence of a generalized Gaussian Birkhoff quadrature formula (GGBQF). Throughout the paper we shall use the notation of [4]. Let $E=(e_{ij})_{i=0,\ j=0}^{n+1\ \ N}$ be an incidence matrix with entries consisting of zeros and ones and satisfying $|E|:=\sum_{i,j} e_{ij}<N+1$ (here we allow a zero row). Let $g(x)$ be a strictly increasing function. It is natural to ask when there exists a GGBQF of the form

$$\int_a^b \sigma(X;x)\,f(x)\,dg=\sum_{e_{ij}=1} a_{ij}f^{(j)}(x_i) \tag{1}$$

which is exact for $\mathbf{P}_N$, the space of all polynomials of degree at most N, where

$$X\colon a=x_0<x_1<\cdots<x_n<x_{n+1}=b \tag{2}$$

and

$$\sigma(X;x)=\operatorname{sgn}\prod_{i=1}^{n}(x-x_i)^{\nu_i} \tag{3}$$

with certain nonnegative integers $\nu_1, ..., \nu_n$.

Many authors have studied this problem. Recently a lot of developments and generalizations have been obtained. For full information on this subject see the introduction and the references in [5]. Here we mention two main results of them.

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THEOREM A [1]. *Let* $U = \operatorname{span}\{u_0, ..., u_N\} \subset C^N[a, b]$ *be an ET-system and* $w \in C[a, b]$ *a positive weight function. Then for* $\nu_0 \geqslant 0$, $\nu_{n+1} \geqslant 0$, $\nu_i > 0$, $i = 1, ..., n$, *such that* $N + 1 = \sum_{i=0}^{n+1} \nu_i$, *there exists a unique set of nodes* X *for which the GGBQF*

$$\int_a^b \sigma(X; x)\, u(x)\, w(x)\, dx = \sum_{i=0}^{n+1} \sum_{j=0}^{\nu_i - 1} a_{ij} u^{(j)}(x_i), \qquad u \in U \tag{4}$$

has the property

$$a_{i, \nu_i - 1} = 0, \qquad i = 1, ..., n. \tag{5}$$

THEOREM B [4, Theorem 10.18; 2, Theorem 1]. *Let* E *be an* $(n+2) \times (N+1)$ *matrix all of whose Hermitian sequences in the interior rows are odd, and all of whose other interior sequences are even. If the number of Hermitian sequences in the interior rows* $1 \leqslant i \leqslant n$ *is* p, *and if* E *satisfies the delayed Pólya conditions with the constant* $\rho = p$:

$$M_k(E) := \sum_{j=0}^{k} \sum_{i=0}^{n+1} e_{ij} \geqslant k + 1 - \rho,$$

$$k = 0, ..., N, \quad M_N(E) = |E| = N + 1 - \rho, \tag{6}$$

then there is a GGBQF (1) *with* $\sigma = 1$ *for all* $f \in \mathbf{P}_N$.

Remark. As we know, a space U of dimension n is said to be a weak Chebyshev space if each $u \in U$ has at most $n - 1$ sign changes. Meanwhile U is nondegenerate if $u \in U$ with the property that $u(x) = 0$ on a nontrivial interval of $[a, b]$ implies $u = 0$. Then in the special case when $\nu_0 = \nu_{n+1} = 0$ and $\nu_1 = \cdots = \nu_n = 1$ the conclusions of Theorem A remain true even if U is a nondegenerate weak Chebyshev space of dimension $N + 1 = n$ according to the following lemma, where such a space is called an IT-space.

LEMMA A [3, Theorem B, Theorem 1]. *Let* $U = \operatorname{span}\{u_1, ..., u_n\}$ *be an IT-space in* $L^1[a, b]$, *the space of all real-valued integrable functions on* $[a, b]$. *Then there exists a unique set of points* (2) *such that*

$$\sum_{i=0}^{n} (-1)^i \int_{x_i}^{x_{i+1}} u(x)\, dx = 0, \qquad u \in U.$$

In this paper we establish a GGBQF for a wide class of incidence matrices which includes the above two matrices.

For a given incidence matrix $E=(e_{ij})_{i=0,\ j=0}^{n+1\ N}$ and each i, $0 \leqslant i \leqslant n+1$, let μ_i denote the smallest index j such that $e_{ij}=0$. Put

$$\Omega(X; x) := \prod_{i=0}^{n+1} (x-x_i)^{\mu_i}, \qquad S(X; x) := \operatorname{sgn} \prod_{i=1}^{n} (x-x_i)^{\mu_i}. \tag{7}$$

The main result in this work is the following

THEOREM 1. *Let an $(n+2)\times(N+1)$ incidence matrix E satisfy the delayed Pólya conditions (6) with the constant $p=p$, $0 \leqslant p \leqslant n$, and contain no odd non-Hermitian sequences in the interior rows $1 \leqslant i \leqslant n$. Then for any prescribed p interior rows i_k, $k=1, ..., p$, there exists a set of nodes (2) such that (1) holds for all $f \in \mathbf{P}_N$, where*

$$v_i = \begin{cases} \mu_i + 1, & i = i_1, ..., i_p, \\ \mu_i, & \textit{otherwise}. \end{cases} \tag{8}$$

The proof of this theorem is a modification of the idea of proof of Theorem 10.18 in [4, p. 148]. The following crucial lemma is an extension of the key lemma in [4, Lemma 10.17, p. 147]. To state this result let Δ be the open simplex consisting of points X satisfying (2) and $\bar{\Delta}$ its closure.

LEMMA. *Let an $(n+2)\times(N+1)$ incidence matrix E satisfy the delayed Pólya conditions (6) with the constant $p=p$, $p \geqslant 1$, and contain no odd non-Hermitian sequences in the interior rows $1 \leqslant i \leqslant n$. Let*

$$G = \{\Omega^{-1}(X; x)\, P(x) : P \in \mathbf{P}_N,\ P^{(j)}(x_i)=0,\ e_{ij}=1,\ e_{ij} \in E\} \tag{9}$$

and

$$W = \{S(X; x)\, P(x) : P \in \mathbf{P}_N,\ P^{(j)}(x_i)=0,\ e_{ij}=1,\ e_{ij} \in E\}. \tag{10}$$

If some of the coordinates of X coincide, E is replaced by its corresponding coalescence. Then

(a) *G is a space of dimension p;*

(b) *G is a Chebyshev space;*

(c) *G has a basis that depends continuously on X.*

Moreover, W is a weak Chebyshev space of dimension p and has a basis that depends continuously on X.

Proof. It is easy to see that G is in fact a set of polynomials. Since by definition

$$W = \{|\Omega(X; x)|\, P(x) : P \in G\},$$

the conclusions of the second part of the lemma follow immediately from the first one. Statements (a) and (c) can be proved by the same arguments as Lemma 10.17 in [4, p. 147]. Now let us show Statement (b).

Let $a < y_1 < \cdots < y_p < b$ be arbitrary but fixed points taken so that $X \cap \{y_1, ..., y_p\} = \varnothing$ and let $Y = X \cup \{y_1, ..., y_p\}$. Let E^* be obtained from E by adding to it p Lagrangian rows. Obviously, E^* satisfies the Pólya conditions (i.e., the delayed Pólya conditions with the constant $p = 0$) and is conservative [4, p. 10]. By the Atkinson–Sharma theorem [4, p. 10] the pair E^*, Y is regular. Now let $P_1(x), ..., P_p(x)$ be the fundamental polynomials of interpolation for this pair corresponding to $e_{ij}^* = 1$ and $e_{ij} = 0$. Put $P = \sum_{k=1}^{p} c_k P_k \neq 0$. Suppose to the contrary that $\Omega^{-1}(X; x)\, P(x)$ has p zeros $a \leqslant z_1 < \cdots < z_p \leqslant b$. If $z_k = x_i$ for some indices k, i, $1 \leqslant k \leqslant p$; $0 \leqslant i \leqslant n+1$, then $P(x_i) = P'(x_i) = \cdots = P^{(\mu_i)}(x_i) = 0$. In this case we add a 1 to the position (i, μ_i). If $z_k \notin X$ then add a new Lagrangian row. Let E' be obtained from E by the above process. Then P is annihilated by E', $Z = X \cup \{z_1, ..., z_p\}$. Since E' is also regular and $|E'| = N+1$, $P = 0$, a contradiction. ∎

Proof of Theorem 1. When $p = 0$ the theorem is trivial, since in this case E is regular and hence the GGBQF (1) with arbitrary X holds for all $f \in \mathbf{P}_N$.

Now let $p > 0$ and let E^* be obtained from E by adding a 1 to the position (i, μ_i) for $i = i_1, ..., i_p$. Then for E^* and arbitrary $X \in \Delta$ the pair E^*, X is regular. Now let $A_{ij}(x) := A_{ij}(X; x)$ be the fundamental polynomials of interpolation for this pair corresponding to $e_{ij}^* = 1$. Then we have

$$f(x) = \sum_{e_{ij}^* = 1} f^{(j)}(x_i)\, A_{ij}(X; x), \qquad f \in \mathbf{P}_N.$$

Hence

$$\int_a^b \sigma(X; x)\, f(x)\, dg = \sum_{e_{ij}^* = 1} a_{ij} f^{(j)}(x_i), \qquad f \in \mathbf{P}_N, \tag{11}$$

where

$$a_{ij} = \int_a^b \sigma(X; x)\, A_{ij}(X; x)\, dg, \qquad e_{ij}^* = 1. \tag{12}$$

To prove Theorem 1 it suffices to show that there exists an X satisfying (2) such that

$$a_{i, \mu_i} = 0, \qquad i = i_1, ..., i_p. \tag{13}$$

To this end, with each $X \in \Delta$ we associate the vector of functions

$$Q_k(X; x) := S(X; x)\, A_{i_k, \mu_{i_k}}(X; x), \qquad k = 1, ..., p, \tag{14}$$

where $S(X; x)$ is defined in (7). Since by the Lemma (14) is a basis of a nondegenerate weak Chebyshev space, according to Lemma A we can define a continuous map $T_2: \bar{\Delta} \to \Delta_0$ (the open simplex consisting of points $a < z_1 < \cdots < z_p < b$) by $Z = T_2 X$ which is the unique solution $a < z_1 < \cdots < z_p < b$ satisfying

$$\int_a^b \left[\operatorname{sgn} \prod_{k=1}^{p} (x - z_k) \right] Q_i(X; x)\, dg = 0, \qquad i = 1, ..., p. \tag{15}$$

As the proof in Theorem 10.18 in [4, p. 148], again we define a continuous map $X = T_1 Z$ of $\bar{\Delta}_0$ into $\bar{\Delta}$ such that

$$x_{i_k} = z_k, \qquad k = 1, ..., p \tag{16}$$

and X satisfies (2).

Thus $T = T_1 T_2$ is a continuous map of $\bar{\Delta}$ into itself. Now applying the Brouwer fixed-point theorem it has a fixed point $X \in \Delta$, i.e.,

$$\int_a^b \left[\operatorname{sgn} \prod_{k=1}^{p} (x - x_{i_k}) \right] Q_i(X; x)\, dg = 0, \qquad i = 1, ..., p,$$

here we use the relations (16). By virtue of (3), (7), (8), and (10) the above formulas become

$$\int_a^b \sigma(X; x)\, A_{i, \mu_i}(X; x)\, dg = 0, \qquad i = i_1, ..., i_p,$$

which by (12) is equivalent to (13). ∎

THEOREM 2. *Let E be defined as in Theorem 1. If $\nu_i - j$ is even for $1 \leqslant i \leqslant n$, then*

$$\operatorname{sgn} a_{ij} = \sigma(X; x_i + 0). \tag{17}$$

Moreover, if $0 \leqslant j \leqslant \nu_0 - 1$, then

$$\operatorname{sgn} a_{0j} = \sigma(X; a + 0) \tag{18}$$

and if $0 \leqslant j \leqslant \nu_{n+1} - 1$, then

$$\operatorname{sgn} a_{n+1, j} = (-1)^j \sigma(X; b - 0). \tag{19}$$

Proof. Let E^* be defined as in the proof of Theorem 1. For the pair (i, j) given in the theorem let E' be obtained from E^* by shifting the 1 in position $(i, v_i - 1)$ into the new position $(0, v_0)$ and by dropping the 1 in position (i, j). Then E' satisfies the conditions of the lemma with $p = 1$. Applying the lemma we conclude that

$$\operatorname{sgn}\left[(x - x_i)^j \prod_{\substack{k=1 \\ k \neq i}}^{n} (x - x_k)^{v_k}\right] P(x)$$

does not change sign in $[a, b]$, where $P \in \mathbf{P}_N$ satisfies

$$P^{(j)}(x_i) = 1, \quad P^{(l)}(x_k) = 0, \quad e'_{kl} = 1, \quad e'_{kl} \in E'.$$

Hence $\sigma(X; x)\, P(x)$ does not change sign in $[a, b]$, because $v_i - j$ is even for $1 \leqslant i \leqslant n$. Since $P^{(j)}(x_i) = 1$, we have that $(x - x_i)^j P(x) > 0$ holds if $|x - x_i| > 0$ small enough. So for these x

$$\begin{aligned}\operatorname{sgn}[\sigma(X; x)\, P(x)] &= \sigma(X; x) \operatorname{sgn}[(x - x_i)^j] \\ &= \begin{cases} \sigma(X; x_i + 0), & 0 \leqslant i \leqslant n \\ (-1)^j \sigma(X; b - 0), & i = n + 1. \end{cases}\end{aligned}$$

Setting $f = P$ in (1) yields

$$a_{ij} = \int_a^b \sigma(X; x)\, P(x)\, dg,$$

which implies (17), (18), and (19) for $1 \leqslant i \leqslant n$, $i = 0$, and $i = n + 1$, respectively. ∎

Using Theorem 1 we can get an extension of Theorem B as follows.

THEOREM 3. *Let E be an $(n+2) \times (N+1)$ matrix all of whose non-Hermitian sequences in the interior rows are even. If the number of odd Hermitian sequences in the interior rows $1 \leqslant i \leqslant n$ is p, $0 \leqslant p \leqslant n$, and if E satisfies the delayed Pólya conditions with the constant $\rho = p$, then there is GGBQF (1) with $\sigma = 1$ for all $f \in \mathbf{P}_N$.*

Proof. It suffices to prescribe the p interior rows i_k, $k = 1, ..., p$ which contain odd Hermitian sequences. ∎

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